Dense locally finite subgroups of automorphism groups of ultraextensive spaces and Vershik's conjecture

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Results in this talk are joint work with Su Gao, Francois Le Maitre and Julien Melleray.

Overview

1. Herwig–Lascar extension theorem
2. Coherent extension and ultraextensive structures
3. Vershik's conjecture and omnigenous groups
4. Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

Herwig–Lascar extension theorem

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One of the first results of this sort was proven by Hrushovski for finite (simple) graphs.

Theorem (Hrushovski, 1992)

Every finite (simple) graph Γ has an extension Γ' such that every partial automorphism of Γ can be extended to an automorphism of Γ'.

Herwig–Lascar extension theorem

Definition

Let C_1, C_2 be two structures in a given finite relational language $\mathcal{L}$. A **partial isomorphism** from C_1 into C_2 is an isomorphism of a finite substructure of C_1 onto a substructure of C_2.
If $C_1 = C_2$, such a map is called a **partial automorphism**.

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Definition

Let $\mathcal{C}$ be a class of $\mathcal{L}-$structures (containing both finite and infinite structures). $\mathcal{C}$ is said to have **the extension property for partial automorphisms (EPPA)** if whenever C_1 and C_2 are structures in $\mathcal{C}$, C_1 is finite, $C_1 \subseteq C_2$, and every partial automorphism of C_1 extends to an automorphism of C_2, then there exist a finite structure C_3 in $\mathcal{C}$ such that every partial automorphism of C_1 extends to an automorphism of C_3.

Herwig–Lascar extension theorem

Definition

If M is an $\mathcal{L}$-structure and $\mathcal{T}$ a set of $\mathcal{L}$-structures, we say that M is $\mathcal{T}$-**free** if there is no structure $T \in \mathcal{T}$ and homomorphism $h : T \to M$.

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Theorem (Herwig–Lascar, 1999)

Let $\mathcal{L}$ be a finite relational language and $\mathcal{T}$ a finite set of finite $\mathcal{L}$-structures. Then the class of all finite $\mathcal{T}$-free $\mathcal{L}$-structures has the EPPA.

Herwig–Lascar extension theorem

Theorem (Solecki, 2005)

Let X be a finite metric space. There exists a finite metric space Y such that X isometrically embeds into Y and every partial isometry of X extends to an (full) isometry of Y.
That is, the class of metric spaces has the EPPA.

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Alternative proofs presented by Pestov (2008) and Vershik (2007), independently. Also, Jan Hubička, Matěj Konečný and Jaroslav Nešetřil recently presented combinatorial proofs of Solecki and HL theorem.

Coherent extensions and ultraextensive structures

Coherent extensions and ultraextensive structures

Definition

Let X be a metric space and $\mathcal{P}$ be the set partial isometries of X. We call (Y, ϕ) an *extension* of X where $X \subseteq Y$, $\phi : \mathcal{P} \to \mathrm{Iso}(Y)$ and for every $p \in \mathcal{P}$, $\phi(p)$ extends p.

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Definition (Solecki)

Let X be a metric space. An extension (Y, ϕ) of X is **strongly coherent** if for every triple (p, q, r) of partial isometries of X such that $p \circ q = r$, we have $\phi(p) \circ \phi(q) = \phi(r)$.

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Theorem (Solecki, 2009)

Let X be a finite metric space. Then, X has a strongly coherent finite extension.

We introduce a weaker notion of coherence.

Definition

Let $X_1 \subseteq X_2$ be metric spaces and (Y_i, ϕ_i) be an extension of X_i for $i = 1, 2$. We say that (Y_1, ϕ_1) and (Y_2, ϕ_2) are **coherent** if

(i) Y_2 extends Y_1,

(ii) $\phi_2(p)$ extends $\phi_1(p)$ for all $p \in \mathcal{P}_{X_1} \subseteq \mathcal{P}_{X_2}$, We introduce a weaker notion of coherence.

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(ii) $\phi_2(p)$ extends $\phi_1(p)$ for all $p \in \mathcal{P}_{X_1} \subseteq \mathcal{P}_{X_2}$, and

(iii) letting $K_i = \phi_i(\mathcal{P}_{X_i}) \subseteq \mathrm{Iso}(Y_i)$ for $i = 1, 2$, and letting $\kappa : K_1 \to K_2$ be the map $\kappa(\phi_1(p)) = \phi_2(p)$ for all $p \in \mathcal{P}_{X_1}$, then κ extends uniquely to a group embedding from $\langle K_1 \rangle$ into $\langle K_2 \rangle$.

Coherent extensions and ultraextensive structures

Definition

A metric space U is **ultraextensive** if

(i) U is ultrahomogeneous, i.e., there is a ϕ such that (U, ϕ) is an extension of U;

(ii) Every finite $X \subseteq U$ has a finite extension (Y, ϕ) where $Y \subseteq U$;

(iii) If $X_1 \subseteq X_2 \subseteq U$ are finite and (Y_1, ϕ_1) is a finite minimal extension of X_1 with $Y_1 \subseteq U$, then there is a finite minimal extension (Y_2, ϕ_2) of X_2 such that $Y_2 \subseteq U$ and (Y_1, ϕ_1) and (Y_2, ϕ_2) are coherent.

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Examples

The following are some examples of ultraextensive spaces: the Urysohn space $\mathbb{U}$, the rational Urysohn space $\mathbb{QU}$ and the random graph $\mathcal{R}$.

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Every countable metric space can be extended to a countable ultraextensive metric space.

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This was proved for $\mathrm{Iso}(\mathbb{QU})$ by Rosendall (2011) and Solecki (2009).

Vershik's conjecture and omnigenous groups

A locally finite group is a group in which every finitely generated subgroup is finite. The following theorem is due to P. Hall.

Theorem (Hall, 1959)

There exists a countable locally finite group $\mathbb{H}$ that is determined up to isomorphism by the following properties:

(A) *any finite group can be embedded in $\mathbb{H}$, and*

(B) *any two isomorphic finite subgroups of $\mathbb{H}$ are conjugate by an element of $\mathbb{H}$.*

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Furthermore, every countable locally finite subgroup can be embedded in $\mathbb{H}$, i.e., $\mathbb{H}$ is a universal countable locally finite group.

Vershik's conjecture and omnigenous groups

Theorem (Hall)

For every triple (G_1, G_2, Ψ), *where* G_1, G_2 *are finite subgroups of* $\mathbb{H}$ *and* $\Psi : G_1 \to G_2$ *is a group isomorphism, there exists* $h \in \mathbb{H}$ *such that for every* $g \in G_1$ *we have* $\Psi(g) = hgh^{-1}$.

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Proposition

Let H be a countable locally finite group with the following property:

(E) for every triple (G_1, G_2, Ψ_1), where $G_1 \leq G_2$ are finite groups and $\Psi_1 : G_1 \to H$ is a group embedding, there exists a group embedding $\Psi_2 : G_2 \to H$ such that $\Psi_2 \upharpoonright G_1 = \Psi_1$.

Then H is isomorphic to $\mathbb{H}$.

Theorem (EGLMM)

The following groups contain Hall's universal countable locally finite group $\mathbb{H}$ as a dense subgroup:

1. *Iso*$(\mathbb{U})$*, the isometry group of the Urysohn space;*
2. *Iso*$(\mathbb{QU})$*, the isometry group of the rational Urysohn space;*
3. *Iso*$(\mathbb{U}_\Delta)$*, the isometry group of the universal* Δ*-metric space, for any distance value set* Δ*;*
4. *Isometry groups of ultrametric Urysohn space;*
5. *Aut*$(\mathcal{R})$*, the automorphism group of the random graph; and*
6. *Aut*(H_n)*, the automorphism group of the universal* K_n*-free graph, for any* $n \geq 3$*.*

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5. *$Aut(\mathcal{R})$, the automorphism group of the random graph; and*
6. *$Aut(H_n)$, the automorphism group of the universal K_n-free graph, for any $n \geq 3$.*

Next, we present two different approaches to proof the above theorem.

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Definition

Let M be a countable $\mathcal{L}$-structure. The **age** of M, $\mathrm{Age}(M)$, is the class of all finitely generated substructures of M (considered up to isomorphism).

Theorem (Fraïssé)

Let $\mathcal{K}$ be a countable (up to isomorphism) class of finite $\mathcal{L}$-structures which has

- *hereditary property: if $A \in \mathcal{K}$ and $B \subseteq A$ then also $B \in \mathcal{K}$,*
- *joint embedding property: for every $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ and embeddings $e\colon A \to C$ and $f\colon B \to C$,*
- *amalgamation property: for every $A, B, C \in \mathcal{K}$ and any embedding $e\colon A \to B$, $f\colon A \to C$, there exists $D \in \mathcal{K}$ and embeddings $g\colon B \to D$ and $h\colon C \to D$ such that $g \circ e = h \circ f$.*

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then there exists a unique (up to isomorphism), ultrahomogeneous countable $\mathcal{L}$-structure M such that $Age(M) = \mathcal{K}$.

Vershik's conjecture and omnigenous groups

Definition

Let $\mathcal{K}$ be the class of all pairs (M, G) such that

- M is a finite $\mathscr{T}$-free $\mathscr{L}$-structure,
- G is a finite group, and
- G acts on M by isomorphisms.

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Theorem (EGLMM)

$\mathcal{K}$ is a Fraïssé class. Furthermore, if (N_∞, H_∞) is the Fraïssé limit of $\mathcal{K}$, then $H_\infty \cong \mathbb{H}$, H_∞ acts faithfully on N_∞, and H_∞ is dense in $Aut(N_\infty)$.

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Let G be a group. We say that G is **omnigenous** if for every finite subgroup $G_1 \leq G$, finite groups $\Gamma_1 \leq \Gamma_2$ and group isomorphism $\Psi_1 : G_1 \cong \Gamma_1$,

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Clearly Hall's group is a countable locally finite omnigenous group. If a class of groups is closed under taking subgroups and homomorphic images and it does not include all finite groups, then it does not include an omnigenous group.

Vershik's conjecture and omnigenous groups

Second approach: Inspired by property (E) of the Hall's group, we define a notion of omnigenous group.

Definition

Let G be a group. We say that G is **omnigenous** if for every finite subgroup $G_1 \leq G$, finite groups $\Gamma_1 \leq \Gamma_2$ and group isomorphism $\Psi_1 : G_1 \cong \Gamma_1$, there is a finite subgroup $G_2 \leq G$ with $G_1 \leq G_2$ and an onto homomorphism $\Psi_2 : G_2 \to \Gamma_2$ such that $\Psi_2 \restriction G_1 = \Psi_1$.

Clearly Hall's group is a countable locally finite omnigenous group. If a class of groups is closed under taking subgroups and homomorphic images and it does not include all finite groups, then it does not include an omnigenous group. Thus, abelian groups, p-groups and solvable groups are not omnigenous.

Theorem (EGLMM)

There are continuum many pairwise nonisomorphic countable universal locally finite omnigenous groups.

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Idea of the proof: Let $P \subseteq \mathbb{P}$ be a subset of prime numbers. We construct a group H_P that is universal for countable locally finite groups; and P is exactly the set of all primes p such that there are order-p elements $\alpha, \beta \in H_P$ that are not conjugate in H_P.

Definition

A **distance value set** is a nonempty subset Δ of the open interval $(0, +\infty)$, such that

$$\forall x, y \in \Delta \quad \min(x + y, \sup(\Delta)) \in \Delta \ .$$

A Δ-**metric space** is a metric space whose nonzero distances belong to Δ.

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Theorem (EGLMM)

Let H be a countable omnigenous locally finite group. Then for any countable distance value set Δ, Iso$(\mathbb{U}_\Delta)$ contains H as a dense subgroup.

Theorem (EGLMM)

There are continuum many pairwise nonisomorphic countable universal locally finite groups each of which can be embedded into Iso$(\mathbb{U}_\Delta)$ *as a dense subgroup.*

Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

Definition

Let G be a group. A **nontrivial mixed identity** in G is a word $w(x) \in G * \langle x \rangle \setminus G$ such that $w(g) = 1$ for all $g \in G$.

Dense locally finite subgroups of $\mathrm{Iso}(\mathbb{U}_\Delta)$

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Let $\Psi_g : G * \langle x \rangle \to G$ be the homomorphism that is identity on G and $\Psi_g(x) = g$. G is MIF iff $\bigcap_{g \in G} \ker(\Psi_g) = \{1\}$.

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Proposition

Let G be a topological group. If G is MIF then any dense subgroup of G is MIF.

Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

Theorem (EGLMM)

1. *For any countable distance value set Δ of cardinality ≥ 2, Iso($\mathbb{U}_\Delta$) is MIF.*

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Examples

1. $\mathbb{H}$ is MIF.

Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

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Examples

1. $\mathbb{H}$ is MIF. (In general, all omnigenous groups are MIF)

Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

Theorem (EGLMM)

1. *For any countable distance value set Δ of cardinality ≥ 2, Iso$(\mathbb{U}_\Delta)$ is MIF.*
2. *The group Iso$(\mathbb{U})$ is MIF.*

Examples

1. $\mathbb{H}$ is MIF. (In general, all omnigenous groups are MIF)
2. Let Alt($\mathbb{N}$) be the group of all finitely supported even permutations on $\mathbb{N}$. Then, Alt($\mathbb{N}$) is **not** MIF (Hull–Osin).

Theorem (EGLMM)

1. *For any countable distance value set Δ of cardinality ≥ 2, $Iso(\mathbb{U}_\Delta)$ is MIF.*
2. *The group $Iso(\mathbb{U})$ is MIF.*

Examples

1. $\mathbb{H}$ is MIF. (In general, all omnigenous groups are MIF)
2. Let $\mathrm{Alt}(\mathbb{N})$ be the group of all finitely supported even permutations on $\mathbb{N}$. Then, $\mathrm{Alt}(\mathbb{N})$ is **not** MIF (Hull–Osin).
3. Since $\mathrm{Alt}(\mathbb{N})$ is dense in S_∞, S_∞ is **not** MIF.

Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

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Corollary

The notion of omnigenous group is strictly stronger than MIF.

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Note that $S_\infty = \mathrm{Iso}(\mathbb{U}_\Delta)$ when $|\Delta| = 1$.

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There is a MIF locally finite p-group.

Since p-groups are not omnigenous, we have:

Corollary

The notion of omnigenous group is strictly stronger than MIF.

Note that $S_\infty = \mathrm{Iso}(\mathbb{U}_\Delta)$ when $|\Delta| = 1$.

Theorem (Hull–Osin)

Let G be a dense subgroup of S_∞. Then, we have the following dichotomy:

- *G is MIF; or*
- *G has a normal subgroup isomorphic to $Alt(\mathbb{N})$.*

Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

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Theorem (EGLMM)

If G is a locally finite group, then G is ∞-MIF iff G is omnigenous.

Dense locally finite subgroups of Iso($\mathbb{U}_\Delta$)

We have the following implications for a countable locally finite group G:

Dense locally finite subgroups of $\mathrm{Iso}(\mathbb{U}_\Delta)$

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$$
\begin{gathered}
G \text{ is } \infty\text{-MIF} \\
\Updownarrow \\
G \text{ is omnigenous} \\
\Downarrow \\
G \text{ is embeddable as a dense subgroup of } \mathrm{Iso}(\mathbb{U}_\Delta) \text{ for } |\Delta| \geq 2 \\
\Downarrow \\
G \text{ is MIF}
\end{gathered}
$$

Dense locally finite subgroups of $\mathrm{Iso}(\mathbb{U}_\Delta)$

Definition

Let Δ be a distance value set. A Δ-*triangle* is a triple (d_1, d_2, d_3) of elements of Δ which satisfies the triangle inequalities, i.e. for all distinct $i, j, k \in \{1, 2, 3\}$, $|d_j - d_k| \leq d_i \leq d_j + d_k$.

Definition

Two distance value sets Δ, Λ are *equivalent* if there exists a bijection $\theta : \Delta \to \Lambda$ such that, for any $(d_1, d_2, d_3) \in \Delta^3$, we have

$$(d_1, d_2, d_3) \text{ is a } \Delta\text{-triangle} \Leftrightarrow (\theta(d_1), \theta(d_2), \theta(d_3)) \text{ is a } \Lambda\text{-triangle.}$$

Theorem (EGLMM)

Let Δ, Λ be two countable distance value sets. The following are equivalent:

1. *Iso*$(\mathbb{U}_\Delta) \cong$ *Iso*$(\mathbb{U}_\Lambda)$ *as topological groups.*
2. *Iso*$(\mathbb{U}_\Delta) \cong$ *Iso*$(\mathbb{U}_\Lambda)$ *as abstract groups.*
3. *There is a homomorphism Iso*$(\mathbb{U}_\Delta) \to$ *Iso*$(\mathbb{U}_\Lambda)$ *with dense image.*
4. Δ *and* Λ *are equivalent.*

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Theorem (EGLMM)

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Theorem (E.–Gao–Li, 2024)

Let Δ, Λ be countable distance value sets with arbitrarily small values. TFAE:

1. Δ *and* Λ *are equivalent.*
2. Δ *is a multiple of* Λ.

For $|\Delta| \geq 2$, the likely answer to the characterization problem for locally finite groups is a condition strictly in between ∞-MIF and MIF, but we do not know that the following question has a negative answer.

Question

For $|\Delta| \geq 2$, are all dense locally finite subgroups of $\mathrm{Iso}(\mathbb{U}_\Delta)$ ∞-MIF?

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Question

For $|\Delta| \geq 2$, are all dense locally finite subgroups of $\text{Iso}(\mathbb{U}_\Delta)$ ∞-MIF?

The second general problem is to explore the possibility of the other extreme, namely to characterize the isomorphism type of $\text{Iso}(\mathbb{U}_\Delta)$ by the isomorphism types of their countable dense subgroups.

Question

If S, T are countable distance value sets that are not equivalent, is there always a dense countable (locally finite) subgroup of one of $\text{Iso}(\mathbb{U}_S)$ and $\text{Iso}(\mathbb{U}_T)$ that cannot be embedded into the other as a dense subgroup?

Thank you!